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Isogroup and 'general' similarity solution of a nonlinear diffusion equation

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Abstract. For a class of one-dimensional nonlinear diffusion equations, where the diffusion coefficient varies as some power of the dependent variable, the invariance group and its Lie algebra are given. The isovector fields which generate the isogroup are then used to derive a 'general' similarity solution. For a particular case the solution can be reduced to a one-parameter group solution which is in full agreement with previously published results.

1. Introduction

Considerable attention has been recently paid to the study of a large variety of physical phenomena governed by evolution equations of a nonlinear nature. Although solutions to such equations can be easily obtained by numerical computation there still remains the need for exact analytical solutions in order to obtain a good understanding of the behaviour of those nonlinear systems. In this latter respect several methods have been advanced for finding those solutions. Among others we have the direct algebraic method of Hereman *et al* (1986), the inverse scattering transform approach of Gardner *et al* (1974), the integration procedure of Hill (1989) and Hirota's (1976) direct method. However, the geometric approach to invariance groups and solution of partial differential equations of Harrison and Estabrook (1971), which emphasise the underlying group structure of the equations, seems to be a more natural and transparent tool of analysis for this class of equations. Here we shall consider the nonlinear one-dimensional diffusion equation

$$\phi_t - (\phi^n \phi_x)_x = 0 \tag{1}$$

where n is a real constant and subscripts denote differentiation with respect to time t and space x. This type of equation has been used by Lonngren and Hirose (1976) to study the expansion into a vacuum of a thermalised electron cloud described by an isothermal Maxwellian distribution. Also, Ahmadi *et al* (1976) investigated the electric penetration into a plasma with current saturating conductivity, whereas Tuck (1976) considered the diffusion of dopants in semiconductors through the substitutional-interstitial mechanism. More recently Anderson and Lisak (1980) modelled the ion temperature diffusion in Tokamak plasmas assuming that the thermal conduction was the dominant loss mechanism. In all the above-mentioned works either a one-parameter group similarity solution was obtained or approximate analytic techniques were required to compute them. Of course the similarity methods used (Ames 1965) yielded exact solutions but restricted to the invariance properties of the equation under a linear group of transformations.

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In this paper we shall establish an exact 'general' similarity solution of (1) for $n \neq 0$ by taking into account not just a single independent generator of the invariance group, but all of them. We shall not consider the $n = 0$ case because it has been dealt with in full extent by Bluman and Cole (1974). The structure of the paper is as follows. In section 2, by means of the Harrison and Estabrook (1971) approach, the generators of the invariance group, i.e. the isovectors, are obtained. This section also includes the corresponding isogroup and its Lie algebra. The previous results are then applied, in section 3, to derive the most general similarity solution directly from group invariance. Finally section 4 is devoted to showing that the result reported by Lonngren and Hirose (1976) is a particular case of the present solution.

2. Isovectors, isogroup and Lie algebra

In the geometric approach of Harrison and Estabrook (1971) for determining the isovectors of a partial differential equation one first recasts the PDE into an equivalent set of differential forms which is a closed ideal I in a q-dimensional space. For the present case such a closed ideal is given by

$$\alpha = \mathrm{d}\phi - u\,\mathrm{d}t - v\,\mathrm{d}x \tag{2}$$

$$\mathrm{d}\alpha = -\mathrm{d}v \wedge \mathrm{d}x - \mathrm{d}u \wedge \mathrm{d}t \tag{3}$$

$$\beta = (u - n\phi^{n-1}v^2)\,\mathrm{d}x \wedge \mathrm{d}t - \phi^n \mathrm{d}v \wedge \mathrm{d}t \tag{4}$$

with

$$u = \phi_t \qquad v = \phi_x. \tag{5}$$

Of course, d stands for the operation of exterior differentiation and $\wedge$ denotes the exterior product operator. For the sake of typographical simplicity, we shall omit the latter in what follows. The set (2)-(4) is the basis of a differential ideal of the Grassman algebra of forms on the five-dimensional manifold spanned by the variables ϕ, u, v, x and t. If we impose independence of x and t and their differentials in the above set we immediately learn that the set is in involution with respect to x and t. This implies, according to Cartan's geometric theory of PDE (Slebodzinski 1970), that there exists a regular integral manifold to be considered as the solution manifold. The infinitesimal symmetries of the close ideal I are isovectors $V = V^k\ \partial/\partial_k$, where the summation convention is assumed and k runs over all the coordinated basis, such that $\mathscr{L}_v I \subset I$ with $\mathscr{L}_v$ denoting the Lie derivative by the vector field V. Then it immediately follows that the isovectors will be given by solving the equations

$$\mathscr{L}_v \alpha = \lambda\alpha \tag{6}$$

$$\mathscr{L}_v \beta = \xi\beta + \omega\alpha + \zeta\,\mathrm{d}\alpha \tag{7}$$

where λ, ξ, ζ and ω are arbitrary 0-, 0-, 0-, and 1-forms respectively. We do not need to include an expression for $\mathrm{d}\alpha$ similar to equations (6) and (7) because $\mathscr{L}_v\,\mathrm{d}\alpha$ is already in the ideal, i.e. $\mathscr{L}_v\,\mathrm{d}\alpha = (\mathrm{d}\lambda)\alpha + \lambda(\mathrm{d}\alpha)$.

To deal with (6) we introduce the 0-form F, defined as

$$F = \langle V, \alpha \rangle \tag{8}$$

where angle brackets denote contraction of the α form with the vector field V. Exterior differentiation of (8) followed by a substitution of (6) yields

$$\mathrm{d}F = \lambda\alpha - \langle V, \mathrm{d}\alpha \rangle. \tag{8'}$$

If we expand F on the basis of 1-forms, namely

$$\mathrm{d}F = F_\phi\,\mathrm{d}\phi + F_u\,\mathrm{d}u + F_v\,\mathrm{d}v + F_x\,\mathrm{d}x + F_t\,\mathrm{d}t$$

in (8′) and substitute (2) and (3) into it, then by equating coefficients of each basis to zero we obtain

$$V^t = -F_u \tag{9}$$

$$V^x = -F_v \tag{10}$$

$$V^v = vF_\phi + F_x \tag{11}$$

$$V^u = uF_\phi + F_t \tag{12}$$

$$V^\phi = F - vF_v - uF_u. \tag{13}$$

In order to obtain the last expression we have made use of (8). We can proceed similarly with (7) by writing, without loss of generality, the 1-form ω as $\omega = A\,\mathrm{d}u + B\,\mathrm{d}v + C\,\mathrm{d}x + D\,\mathrm{d}t$ with A, B, C and D arbitrary, and once more by equating the coefficients of all the 2-forms to zero we end up with a system of nine equations from which ξ, ζ, A, B, C and D can be eliminated to yield

$$V^t_u = 0 \tag{14}$$

$$\phi^n(V^v_u - vV^t_\phi - V^t_x) - (u - n\phi^{n-1}v^2)(V^x_u + V^t_v) = 0 \tag{15}$$

and

$$\begin{aligned} &v\phi^n V^v_\phi + n(n-1)\phi^{n-2}v^2V^\phi - V^u + 2n\phi^{n-1}vV^v \\ &-(u - n\phi^{n-1}v^2)(vV^x_\phi + V^x_x - V^v_v - n\phi^{-1}V^\phi) \\ &-(u - n\phi^{n-1}v^2)^2\phi^{-n}V^x_v + \phi^n V^v_x = 0. \end{aligned} \tag{16}$$

The above set of PDE can be solved easily for the components of V by a straightforward integration, namely

$$V^t = \delta_1 + \delta_3 t \tag{17}$$

$$V^x = \delta_2 + \delta_4 x \tag{18}$$

$$V^v = \frac{1}{n}[(2-n)\delta_4 - \delta_3]v \tag{19}$$

$$V^u = \frac{1}{n}[2\delta_4 - (n+1)\delta_3]u \tag{20}$$

$$V^\phi = \frac{1}{n}[2\delta_4 - \delta_3]\phi \tag{21}$$

where the δ_i $(i = 1, 2, 3, 4)$ are constants. From (17)-(21) and either (11)-(13) or (8), we obtain for the 0-form F,

$$F = \frac{1}{n}(2\delta_4 - \delta_3)\phi - (\delta_2 + \delta_4 x)v - (\delta_1 + \delta_3 t)u. \tag{22}$$

Thus we have a four-parameter invariance group for (1) with $n \neq 0$. Each of the independent generators of the isogroup are obtained by setting all the parameters but one equal to zero. The complete results are presented in table 1.

Table 1. Invariance group of $\Phi_t = (\Phi^n \Phi_x)_x (n \neq 0)$.

δ_i	V^t	V^x	V^Φ	V^u	V^v
δ_1	1	0	0	0	0
δ_2	0	1	0	0	0
δ_3	t	0	$-\frac{1}{n}\Phi$	$-\left(\frac{n+1}{n}\right)u$	$-\frac{1}{n}v$
δ_4	0	X	$\frac{2}{n}\Phi$	$\frac{2}{n}u$	$\left(\frac{2-n}{n}\right)v$

In this particular instance a physical description of the independent generators is feasible. Namely, a time translation ($\delta_1 = 1$), a space translation ($\delta_2 = 1$), a t and ϕ scale change ($\delta_3 = 1$) and an x and ϕ scale change ($\delta_4 = 1$). If we denote by V_M ($M = 1, 2, 3, 4,$) the different isovector fields, then it immediately follows from table 1 that we have

$$V_1 = \frac{\partial}{\partial t} \tag{23}$$

$$V_2 = \frac{\partial}{\partial x} \tag{24}$$

$$V_3 = -\frac{1}{n}\left(\phi\frac{\partial}{\partial\phi} + (n+1)u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) + t\frac{\partial}{\partial t} \tag{25}$$

$$V_4 = \frac{1}{n}\left(2\phi\frac{\partial}{\partial\phi} + 2u\frac{\partial}{\partial u} + (2-n)v\frac{\partial}{\partial v}\right) + x\frac{\partial}{\partial x}. \tag{26}$$

With the help of (23)–(26) the structure constants of the Lie algebra, f^L_{MN}, become $f^1_{13} = -f^1_{31} = 1$, $f^2_{24} = -f^2_{42} = 1$ and zero otherwise. Therefore the algebra is not Abelian. We also have that V_1 and V_2 are a proper invariant subalgebra. Hence the Lie algebra of the isogroup is neither a simple Lie algebra nor a semisimple Lie algebra.

3. 'General' similarity solution

To search for a solution of the similarity type we take advantage of the fact that the Lie algebra of the isogroup has a proper invariant subalgebra and we can then augment the ideal of forms and impose that the augmented forms be annulled in the solution manifold as well as the original ideal of forms (Harrison and Estabrook 1971). One way to augment the ideal is by contracting α, $d\alpha$ or β with V. Let us choose the contraction $\langle V, \alpha\rangle$ which has already been introduced in section 2 (see (8)). As a result of the involution of α, $d\alpha$ and β with respect to x and t these forms are annulled on the solution manifold, then in order to satisfy that the augmented ideal be also annulled on the solution manifold, we require that (22) be equal to zero, i.e.

$$\phi = n(\theta_2 + \theta_4 x)\phi_x + n(\theta_1 + \theta_3 t)\phi_t \tag{27}$$

where $\theta_i = \delta_i/(2\delta_4 - \delta_3)$ with $2\delta_4 - \delta_3 \neq 0$.

This last equation is a quasilinear PDE which can be easily integrated by the method of characteristics (Courant and Hilbert 1962), i.e.

$$\tau = \frac{\theta_2 + \theta_4 x}{(\theta_1 + \theta_3 t)^{\theta_4/\theta_3}} \tag{28}$$

$$\phi(x, t) = G(\tau)(\theta_1 + \theta_3 t)^{1/n\theta_3} \tag{29}$$

with $G(\tau)$ satisfying the equation

$$G_{\tau\tau} + nG^{-1}G_\tau^2 + \frac{\tau}{\theta_4} G^{-n} G_\tau - \frac{1}{n\theta_4^2} G^{1-n} = 0. \tag{30}$$

Equation (30) results from a substitution of (29) into (1) and then a change of variables from x and t to τ by means of (28).

Therefore (29) is a four-parameter group similarity solution of (1) for $n \neq 0$ as long as $G(\tau)$ satisfies (30). Here τ plays the role of similarity variable. We refer to (29) as a 'general' similarity solution in the sense that is the most general similarity solution obtained directly from group invariance.

4. The $N = -1$ case

In this section we shall show how to recover the Lonngren and Hirose (1976) solution for (1) with $n = -1$ as a particular case of the present one. For the sake of simplicity we shall set $\delta_3 = \delta_4 \neq 0$, which is in agreement with the condition $2\delta_4 - \delta_3 \neq 0$, then (28)-(30) reduce to

$$\tau = \frac{\theta_2 + x}{\theta_1 + \tau} \tag{31}$$

$$\phi(x, t) = G(\tau)(\theta_1 + t)^{-1} \tag{32}$$

$$G_{\tau\tau} - G^{-1}G_\tau^2 + \tau G G_\tau + G^2 = 0. \tag{33}$$

Now we look for a solution of (33) by using a generating function of Lorentz type (Wilhelmsson 1984, Anderson *et al* 1984), i.e.

$$G(\tau) = \frac{a}{p + \tau^2} \tag{34}$$

with a and p parameters to be determined from (33). By substitution of (34) into (33) and equating to zero the coefficients of succesive powers of τ, we obtain $a = 2$ and p any real constant. Then it immediately follows that

$$\phi(x, t) = \left\{ \frac{(\theta_1 + t)}{2} \left[p + \left(\frac{\theta_2 + x}{\theta_1 + t} \right)^2 \right] \right\}^{-1}. \tag{35}$$

As the Lonngren and Hirose (1976) method of solution presupposes only a one-parameter group of transformations and we have already assumed that $\delta_3 = \delta_4 \neq 0$, then the reduction of (35) to a one-parameter group solution is possible if we set $\delta_1 = \delta_2 = 0$ (i.e. $\theta_1 = \theta_2 = 0$). Taking into account the above considerations, (35) becomes

$$\phi(x, t) = \left\{ \frac{t}{2} \left[p + \left(\frac{x}{t} \right)^2 \right] \right\}^{-1}$$

which is the same solution previously reported by the authors referred to earlier.

A final remark regarding Hill's results (Hill 1980) and the present formalism is in order. The choice of similarity variables is mainly dictated by whether the resulting equation is amenable to an easy solution or not. Hill's paper has made a different choice of similarity variables as compared with the ones used in this paper but equally valid. In fact Hill's equations (1.4) and (3.1) can be derived from our (27) by a simple change of variables and a reduction to a one-parameter group description. In order to conform (27) to Hill's notation, we rewrite it as

$$c = m(\theta_2 + \theta_4 x)c_x + m(\theta_1 + \theta_3 t)c_t. \tag{36}$$

To reduce the above expression to a one-parameter group we set $\theta_1 = 0$, $\theta_2 = 0$ and define $\theta_3 = (1/\lambda)$ and $\theta_4 = (1+\lambda)/2\lambda$, then the Lagrange subsidiary equations of (36) become

$$\frac{\mathrm{d}c}{c} = \left(\frac{2\lambda}{m(1+\lambda)}\right)\frac{\mathrm{d}x}{x} = \left(\frac{\lambda}{m}\right)\frac{\mathrm{d}t}{t}.$$

A straightforward integration of the first and second equalities of the above expression yield Hill's similarity variables (1.4) and then his (3.1) follows immediately.

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